Velocity width of the resonant domain in wave-particle interaction

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Wave-particle interaction is a ubiquitous physical mechanism exhibiting locality in velocity space. A single-wave Hamiltonian provides a rich model by which to study the self-consistent interaction between one electrostatic wave and N quasiresonant particles. For the simplest nonintegrable Hamiltonian coupling two particles to one wave, we analytically derive the particle velocity borders separating quasi-integrable motions from chaotic ones. These estimates are fully retrieved through computation of the largest Lyapunov exponent. For the large-N particle self-consistent case, we numerically investigate the localization of stochasticity in velocity space and test a qualitative estimate of the borders of chaos.

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I. PHYSICAL CONTEXT AND INTRODUCTION TO THE PROBLEM

Plasmas are, contrary to ordinary gases, composed of *charged* particles. They are thus sensitive to electromagnetic fields and, moreover, behave similarly to sources generating their own self-consistent fields. These fields may be Fourier decomposed into waves that propagate at specific phase velocities and interact with the so-called resonant particles whose velocities are close to the phase velocities. This phenomenon is called *wave-particle interaction* (see, e.g., [1]). It is ubiquitous in hot plasma physics and plays a major role in controlled thermonuclear fusion physics, in laser plasma interaction, in astrophysics, as well as in semiconductors and electronic devices [2]. One of its features is that it exhibits a *locality* in the velocity space: since all plasma charged particles do not interact in the same manner with the self-consistent fields, a fluid description would fail to describe it and a kinetic description is required. To illustrate this, let us consider Langmuir wave-particle interaction. This is the phenomenon describing the (collisionless) interaction of long-wavelength electrostatic waves with the resonant particles, namely, electrons whose velocities are close to the phase velocities of the modes. Ions having masses far larger than electron masses are assumed fixed and provide a neutralizing background. The usual description of Langmuir wave-particle interaction involves then the coupled kinetic set of Vlasov-Poisson equations for the electron distribution function. In this framework, the locality of wave-particle interaction is evidenced for instance by Landau damping. As emphasized mathematically by Landau's rigorous calculation [3], this phenomenon is governed by resonant particles.

In order to describe precisely this Langmuir wave-particle interaction and take into account these locality properties, an alternative N-body Hamiltonian model has been proposed [4]. This model results from a reduction of the dynamics of the whole plasma. Basically, the bulk particles (whose velocities are roughly smaller than the thermal velocity) only participate in the dynamics via a small subset of collective modes, the Langmuir (long-wavelength) modes of longitudinal oscillation around their center guide positions. These modes are described by action-angle variables (I_j,θ_j) with $1\leq j\leq M$ for the M waves. In the absence of resonant particles, they oscillate with constant angular frequencies $d\theta_j/dt=\omega_{0j}$, which, according to the Bohm-Gross dispersion relation [5], are approximately equal to the plasma frequency ω_P for long-wavelength modes [4]. These waves strongly interact with those plasma particles in the tails of the distribution function having velocities v close to ω_{0j}/k_j. The coupling is controlled by the small parameter η that denotes the ratio of the tail density over the bulk plasma density. The dynamics of N identical quasiresonant particles moving on the interval of length L with periodic boundary conditions, with unit mass and charge, and, respectively, position x_r and momentum p_r, interacting with M waves with wave numbers $k_j=j2\pi/L$, derives then from the Hamiltonian

$$H=\sum_{l=1}^{N}\frac{p_l^2}{2}+\sum_{j=1}^{M}\omega_{0j}I_j-N^{-1/2}\sum_{l=1}^{N}\sum_{j=1}^{M}\sqrt{2\eta I_j}\times\cos(k_jx_l-\theta_j). \tag{1}$$

However, the discrimination between quasiresonant "relevant" particles and particles that should be included in the plasma bulk, because they participate only marginally in the wave-particle interaction, is presently rather loose and qualitative. For the simplest and most paradigmatic case of one single wave, it remains to determine in this self-consistent model, a picture of the various amounts of stochasticity in the velocity space and to try to estimate the width of the stochastic sea developing around the wave resonance. The study of the localization of chaos in the phase space is also crucial to determining which particles should be taken into account if one wishes to derive relevant Gibbs statistical mechanics predictions [6] (that agree with temporal averages due to a sufficient mixing in phase space). Our aim in this paper is to bring a semiquantitative resolve to this issue.

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II. THE SINGLE-WAVE HAMILTONIAN

Let us introduce the self-consistent Hamiltonian coupling N particles and one wave (with a unit wave number)

$$H=\sum_{l=1}^{N} \frac{p_l^2}{2}+\omega_0 I-\sqrt{2\eta I/N}\sum_{l=1}^{N} \cos(x_l-\theta). \tag{2}$$

In order to fix notations, we put the wave number at $j=1$ and the spatial period at $L=2\pi$. This single-wave Hamiltonian was first formulated as a simplified model to treat the instability due to a weak cold electron beam in a plasma, assuming a fixed ionic neutralizing background [7,8]. Yet recently, different studies have extended the regime of application of the single-wave model to a much larger class of instabilities [9], derived it in a generic manner from different contexts [10], or proved it could model various physical phenomena (e.g., Compton free-electron laser amplification [11]). There exists a large variety of similar Hamiltonian models (for instance, see reference [12], for the description of Alfvén-wave particle interaction).

It is useful to write down the equation of motion of any particle l as

$$\ddot{x}_l=-\sqrt{2\eta I/N}\sin(x_l-\theta). \tag{3}$$

That is the equation of a pendulum evolving in the self-consistent field with strength $\sqrt{2\eta I/N}$ and angle θ (whose temporal evolutions depend on the evolution of all the particles).

A Galilean transformation enables us to put the system in the reference frame of the wave. Actually, considering the generating function $F_1(\mathbf{x},\theta,\overline{\mathbf{p}},\overline{I},t)=\Sigma_{l=1}^{N}(x_l-\omega_0 t)\overline{p_l}+(\theta-\omega_0 t)\overline{I}+\Sigma_{l=1}^{N}\omega_0 x_l-N\omega_0^2 t/2$, the new Hamiltonian becomes

$$\overline{H}(\overline{\mathbf{p}},\overline{I},\overline{\mathbf{x}},\overline{\theta})=H+\frac{\partial F_1}{\partial t}=\sum_{l=1}^{N} \frac{1}{2}\overline{p}_l^2-\sqrt{2\eta\overline{I}/N}\sum_{l=1}^{N} \cos(\overline{x_l}-\overline{\theta}). \tag{4}$$

It is easy to check that the total momentum

$$\overline{P}=\sum_{l=1}^{N} \overline{p_l}+\overline{I} \tag{5}$$

is conserved by the dynamics obtained from Eq. (4). This enables us to define a new generator as $F_2(\overline{\mathbf{x}},\overline{\theta},\mathbf{p},\overline{P})=\overline{P}\,\overline{\theta}+\Sigma_{l=1}^{N}p_l(\overline{x_l}-\overline{\theta})$. The new coordinates conjugated to the p_l's are given by $q_l=\partial F_2/\partial p_l=\overline{x_l}-\overline{\theta}$. The final Hamiltonian, emphasizing that only N degrees of freedom are effective, is given in the compact form

$$H(\mathbf{p},\mathbf{q})=\sum_{l=1}^{N}\left[\frac{p_l^2}{2}-\sqrt{2\eta/N}\left(\overline{P}-\sum_{l=1}^{N} p_l\right)^{1/2}\cos q_l\right]. \tag{6}$$

The wave-particle self-consistency manifests itself through a coupling potential whose strength depends, in a mean-field way, on all particle velocities. This interaction is typical of wave-particle interaction and we shall investigate how this particular form of coupling affects wave-particle locality properties.

In this new form (6), it is clear that the self-consistent interaction of two particles and one wave is the simplest nonintegrable situation for the model. We shall first study in this case how self-consistency acts on velocity locality. This case will provide a good illustration of how a velocity-dependent coupling affects the localization of chaos in velocity space. We shall see in Sec. IV that this study can, with some restrictions, be relevant to the general problem of the interaction of a large number N of particles with a single wave. In that case, a numerical study is appropriate to estimate the velocity domain in which particles play an active role in wave-particle interaction.

III. VELOCITY BORDERS OF CHAOS FOR THE COUPLING OF TWO PARTICLES AND ONE WAVE

A. Local reduction to a one-and-a-half degrees of freedom Hamiltonian system

The effective Hamiltonian in the reference frame of the wave is

$$H(\mathbf{p},\mathbf{q})=\frac{p_1^2}{2}+\frac{p_2^2}{2}-\sqrt{\eta}(\overline{P}-p_1-p_2)^{1/2}(\cos q_1+\cos q_2), \tag{7}$$

where one recalls that η is a small parameter. Therefore, we shall consider the system as close to integrability. We denote by $H_0(p_1,p_2)=p_1^2/2+p_2^2/2$ the energy of the unperturbed system and use the method developed by Escande and Doveil [13,14] to show that the system is locally reducible to a one-and-a-half degrees of freedom Hamiltonian system.

This reduction operates in the vicinity of a reference pair of actions that we denote by $\mathbf{p_r}=(p_{1r},p_{2r})$. If the system were integrable (without the perturbative potential part), both particles would move ballistically. When the perturbation is on, the effective dynamics can be proved to be reducible to one-and-a-half degrees of freedom as long as the velocities of the two particles do not depart too much from p_{1r} and p_{2r}, which will be the case if they are far enough from the wave resonance that their motion is only slightly affected by the wave. The key point explaining the local reduction of the dynamics to one-and-a-half degrees of freedom is that, in the vicinity of $\mathbf{p_r}$, the energy circle $H_0(p_{1r},p_{2r})\equiv E_r=H_0(p_1,p_2)$ is similar to a parabolic branch. In order to move the system to the reference frame with origin $\mathbf{p}_r$ oriented according to the principal axes of the parabola, with vectors $\mathbf{\Omega}_r$ and $\mathbf{r}$ defined through

$$\mathbf{\Omega_r}=\left.\frac{\partial H_0}{\partial \mathbf{p}}\right|_{\mathbf{p}=\mathbf{p_r}}=\begin{pmatrix} p_{1r}\\ p_{2r}\end{pmatrix}=\mathbf{p_r} \tag{8}$$

and $\mathbf{r}\cdot\mathbf{\Omega_r}=0$ with $\|\mathbf{p}_r\|=\|\mathbf{r}\|=(2E_r)^{1/2}$ so that

$$\mathbf{r}=\begin{pmatrix} p_{2r}\\ -p_{1r}\end{pmatrix}, \tag{9}$$

one has to operate one translation in momentum space, followed by one rotation of the initial Cartesian reference frame, that is, a unitary transform. This defines a canonical transform of the original variables $(\mathbf{p},\mathbf{q})$ towards the new variables $(\mathbf{p}',\mathbf{q}')$ with the generating function

$$F(\mathbf{q},\mathbf{p}')=\mathbf{q}\cdot(\mathbf{p_r}+p_1'\mathbf{r}+p_2'\mathbf{\Omega_r}). \tag{10}$$

This gives

$$\mathbf{p}=\frac{\partial F}{\partial \mathbf{q}}=\mathbf{p_r}+p_1'\mathbf{r}+p_2'\mathbf{\Omega_r},$$

$$q_1'=\frac{\partial F}{\partial p_1'}=p_{2r}q_1-p_{1r}q_2,$$

$$q_2'=\frac{\partial F}{\partial p_2'}=p_{1r}q_1+p_{2r}q_2.$$

The new Hamiltonian then reads

$$\begin{aligned}\tilde{H}(\mathbf{p}',\mathbf{q}')=&E_r+E_r(p_1'^2+2p_2'+p_2'^2)-\sqrt{\eta}[\overline{P}-p_{1r}\\&\times(1-p_1'+p_2')-p_{2r}(1+p_1'+p_2')]^{1/2}\\&\times\left[\cos\left(\frac{p_{2r}q_1'+p_{1r}q_2'}{2E_r}\right)+\cos\left(\frac{p_{1r}q_1'-p_{2r}q_2'}{2E_r}\right)\right].\end{aligned}$$

Locally, that is, in the vicinity of the reference action $\mathbf{p_r}$, one can neglect $p_2'^2$ compared to $2p_2'$, as well as $\pm p_1'+p_2'$ compared to 1 in the potential strength term. Under these conditions, q_2' depends linearly on time, since its derivative $\dot{q}_2'$ is then equal to $2E_r$, that is to a constant. Therefore, on the restricted phase space $\mathcal{S}_1$ associated with the first particle, the dynamics corresponds to a one-and-a-half degrees of freedom Hamiltonian one, that is formally the dynamics exhibited by the two-wave paradigmatic Hamiltonian defined in Ref. [14]. Actually, by changing time origin through $t''=t-t_0$, one obtains $q_2'=2E_r t''$. Moreover, considering the canonical transform defined by the generator

$$F_3(q_1',p_1'')=\frac{1}{2E_r}q_1'p_1''-(E_r+2E_r p_2')t'' \tag{11}$$

gives

$$q_1''=\frac{\partial F_3}{\partial p_1''}=\frac{1}{2E_r}q_1',$$

$$p_1'=\frac{\partial F_3}{\partial q_1'}=\frac{1}{2E_r}p_1''.$$

The expression of the new Hamiltonian is then

$$\begin{aligned}H''(q_1'',p_1'',t'')=&H+\frac{\partial F_3}{\partial t''}\\=&\frac{1}{4E_r}p_1''^2-\sqrt{\eta}(\overline{P}-p_{1r}-p_{2r})^{1/2}\\&\times[\cos(p_{2r}q_1''+p_{1r}t'')\\&+\cos(p_{1r}q_1''-p_{2r}t'')].\end{aligned}$$

A change of scale (noncanonical transform) enables us to obtain the normalized expression of the paradigm Hamiltonian with two waves. Actually, putting $y_1:=p_1''/(2E_r)$, the couple (q_1'',y_1) evolves then in $\mathcal{S}_1$ according to

$$\dot{q}_1''=\frac{\partial K}{\partial y_1} \tag{12}$$

and

$$\dot{y}_1=-\frac{\partial K}{\partial q_1''}, \tag{13}$$

with

$$\begin{aligned}K=&\frac{1}{2}y_1^2-\frac{\sqrt{\eta}(\overline{P}-p_{1r}-p_{2r})^{1/2}}{2E_r}[\cos(p_{2r}q_1''+p_{1r}t'')\\&+\cos(p_{1r}q_1''-p_{2r}t'')].\end{aligned} \tag{14}$$

One identifies here two resonances whose locations are given by the condition of stationary phase: the first one is centered on the momentum $-p_{1r}/p_{2r}$ (phase velocity of the first wave if it is alone) and the second one on p_{2r}/p_{1r} (phase velocity of the second wave if it is alone).

B. Stochasticity in the reduced system

Let us now recall the definition of Chirikov stochasticity parameter s [15]. It is defined by the ratio of the sum $\delta_1+\delta_2$ of the maximum half widths of two neighboring resonances over the distance in velocity space l between both resonances in $\mathcal{S}_1$. When $s\ll 1$, the two resonances are well separated from each other by a layer of invariant curves [Kol'mogorov-Arnol'd-Moser (KAM) tori]. According to the initial conditions, a particle in model (14) will either be trapped in the first or second wave or move on a passing KAM torus. The qualitative overlapping criterion $s\gtrsim 1$, proposed by Chirikov [15], corresponds to the onset of strong chaos. A treatment based on a renormalization group theory [13] has enabled us to fix a quantitative threshold s_c, roughly equal to 0.7, for the disappearance of the last KAM torus between the resonances that marks the onset of this large-scale chaos.

For Eq. (14), one gets $\delta_1=\delta_2=2\eta^{1/4}(\overline{P}-p_{1r}-p_{2r})^{1/4}(2E_r)^{-(1/2)}$. In addition, the two resonances are separated in the velocity space of $\mathcal{S}_1$ by a length $l=|p_{2r}/p_{1r}+p_{1r}/p_{2r}|$. The stochasticity parameter is therefore equal to

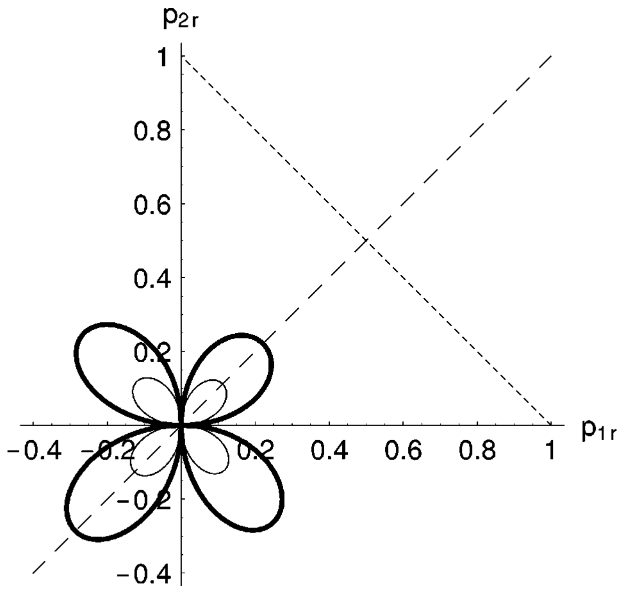

FIG. 1. Curves of isostochasticity $s(p_{1r},p_{2r})=0.7$ (bold line) and $s(p_{1r},p_{2r})=1.5$ (thin line) for the approximated system coupling two particles to one wave. The integrability line $p_{1r}+p_{2r}=\overline{P}$ has been drawn. Values $\eta=2.56\times10^{-4}$ and $\overline{P}=1$ have been used.

$$s=s(p_{1r},p_{2r})=4\,\eta^{1/4}(\overline{P}-p_{1r}-p_{2r})^{1/4}\frac{|p_{1r}p_{2r}|}{(p_{1r}^2+p_{2r}^2)^{3/2}}. \tag{15}$$

Figure 1 represents different curves satisfying $s(p_{1r},p_{2r})=\text{const}$. The external thick curve corresponds to the critical value $s_c\sim0.7$. Along the bisectrix, the chaos increases as one gets closer to the origin. It is important to note that this expression does not involve the positions of the particles but only their reference velocities, as a consequence of the Hamiltonian form (6).

Let us now interpret this figure by starting from a situation of quasi-integrability for H. It is obvious that $\overline{P}-p_{1r}-p_{2r}$ must be positive, and that around the reference configuration $\overline{P}-p_{1r}-p_{2r}=0$, the motion of both particles is quasiballistic: this fixes the "integrability border" drawn in Fig. 1. In this extreme case, the dynamics will be even reducible to a one single degree of freedom (i.e., integrable) system. It is clear that if one provides initially one particle with a large momentum ($p_{1r}\lesssim\overline{P}$), then the width of wave resonance ("cat's eye") will be small: the reference resonance half width will actually be $2\omega_{Br}=2\,\eta^{1/4}(\overline{P}-p_{1r}-p_{2r})^{1/4}$ and will be small compared to p_{1r}. The motion of the first particle will then almost coincide with a KAM torus. Also, if one moves along the first separatrix in the opposite direction, starting from reference momenta having large (compared to $\overline{P}$) negative values, then p_{1r} is large in front of the trapping angular frequency $\eta^{1/4}(\overline{P}-p_{1r}-p_{2r})^{1/4}$, that is, of order $\eta^{1/4}(-p_{1r})^{1/4}$. This situation corresponds also to an almost integrable motion of the particles. It is interesting to note that Eq. (15) is obviously symmetric with respect to the transformation $(p_{1r},p_{2r})\mapsto(p_{2r},p_{1r})$, that is, the simple inversion of the two particles, but that it is not invariant by the transformations $p_{1r}\mapsto-p_{1r}$ and $p_{2r}\mapsto-p_{2r}$ nor by their composition. On the first bisectrix, there is actually a smaller velocity range of strong chaos for positive than for negative velocity values for $\overline{P}$ positive. According to expression (15), the Chirikov parameter should diverge as $(p_{1r},p_{2r})\rightarrow(0,0)$. As Eq. (15) is independent of particle position, this simply signals the existence of the resonance hyperbolic points at zero momentum. These dominate the stochastic behavior of the system on the line of zero momentum.

It is important to note that this reduction of dynamics is only locally valid: with given reference momenta (p_{1r},p_{2r}) at time t_0, this reduction will typically be valid up to some time $t_0+\tau$ at which the hypothesis $p_2'^2\ll2p_2'$ breaks down. At that time, a new reduction treatment could be made for a new reference couple. This method has been used by Escande *et al.* [16] for a nearest-neighbors Hamiltonian model of rotators. They showed that the existence of at least one such 1.5 degrees of freedom's active resonance was sufficient to sweep phase space and to provide good relaxation properties of the microscopic microcanonical dynamics towards Gibbs equilibrium.

C. Largest Lyapunov exponent

The largest Lyapunov exponent (LLE) [17] measures the average degree of chaoticity exhibited by a generic nonintegrable Hamiltonian system.

Consider the equations of motion associated with a given N-dimensional dynamical system defined as

$$\dot{x}_i=X^i(x^1,\ldots,x^N), \tag{16}$$

where $1\leqslant i\leqslant N$, and such that $\mathbf{x}(t=0)\equiv\mathbf{x}_0$, and consider the evolution of a phase-space point initially arbitrarily close by $\delta\mathbf{x}_0$ to the latter one. Then linearizing (16) around the reference trajectory $\mathbf{x}_R(t)$ gives the usual tangent equations

$$\delta\dot{x}_i=J_k^i[\mathbf{x}_R(t)]\,\delta x_k, \tag{17}$$

where the $J_k^i[\mathbf{x}_R(t)]$ stand for the Jacobian matrix time-dependent elements related to the X^i.

Then the limit

$$\lim_{t\rightarrow\infty}\frac{1}{t}\ln\frac{\|\delta\mathbf{x}_t\|}{\|\delta\mathbf{x}_0\|} \tag{18}$$

is proved to exist (by the Oseledec theorem) and is, by definition, equal to the largest Lyapunov exponent, denoted λ_1, whose value depends on the ergodic component to which the initial reference condition belongs. Since generally the Jacobian matrices (17) at different times do not commute, one cannot find a common system of eigenvectors, so that usual estimations of the LLE rely on numerical simulations that integrate Eq. (17) around given reference trajectories, with a few remarkable analytical estimations [18].

For our purpose, we shall only compute the positive Lyapunov exponent associated with the effective one-and-a-half degrees of freedom dynamics appearing in Eq. (3) and use the method developed by Habib and Ryne [19]. This method based on the use of symplectic matrices only requires that the linearized dynamics be Hamiltonian and avoids reorthonormalization processes. Linearizing the equation of motion (3) of a test particle around a reference trajectory $x_R(t)$ yields the system

$$\dot{\delta x} = \delta p, \tag{19}$$

$$\dot{\delta p} = -\varepsilon\beta\sqrt{2I(t)}\cos[x_R(t)-\theta(t)]\,\delta x.$$

These equations derive from a quadratic Hamiltonian with 1.5 degrees of freedom of the form $H(\delta p,\delta x,t) := 1/2(s_{11}\delta x^2 + s_{12}\delta x\,\delta p + s_{21}\delta p\,\delta x + s_{22}\delta p^2)$ with $s_{12}=s_{21}=0$ and

$$s_{11} = \varepsilon\beta\sqrt{2I(t)}\cos[x_R(t)-\theta(t)],$$

$$s_{22}=1.$$

In order to obtain λ_1, it is sufficient to solve instead of Eqs. (19) the first-order differential equation system

$$\frac{d\mu}{dt} = \frac{1}{2}(s_{22}-s_{11})\cos a,$$

$$\frac{da}{dt} = s_{11}+s_{22}-(s_{22}-s_{11})\sin a \coth\mu \tag{20}$$

simultaneously with the reference trajectory $x_R(t)$, wave intensity $I(t)$, and phase $\theta(t)$ that are extracted from the parallel fourth-order symplectic integration [20] of the full self-consistent Hamiltonian system (2).

The LLE is then given by

$$\lambda_1 = \lim_{t\to\infty}\frac{\mu(t)}{t}. \tag{21}$$

Let us note that, in numerical simulations, we shall obviously only compute finite- (large)-time LLE.

Through the LLE, which is an indicator of the chaos experienced by microscopic dynamics, we look for a dynamical signature of the onset of strong stochasticity in the system coupling two particles and one wave. For the initial conditions, we restrict ourselves to the symmetric case where both particles have initially the same velocity $p_0=p_{10}=p_{20}$ with arbitrary positions. This corresponds to investigating the stochasticity along the bisectrix $p_{1r}=p_{2r}$ in Fig. 1. Actually, assume first that the initial velocity of the particles is far away from the origin. Then the motion of the particles will be almost ballistic and the local reduction to Eq. (14) will be valid for almost infinite time with $p_{1r}=p_{2r}=p_0$. This will correspond to a vanishing LLE calculated for the reference trajectory given by an initial arbitrary position and an initial velocity $p_0=p_{10}=p_{20}$. As the initial velocity of both particles gets closer to the origin, we expect a continuous increase of chaos in the system so that the condition of reduction, around this reference initial condition, to a one-and-a-half degrees of freedom Hamiltonian $p_2'^2 \ll 2p_2'$, remains valid only during shorter and shorter periods of time. Figure 2 presents the LLE, averaged over 100 different initial positions for both particles, computed through Eqs. (20) and (21) around the trajectory of one particle with arbitrary initial position and initial velocity $p_0=p_{10}=p_{20}$. We can check that the integrable initial condition $p_0=\overline{P}/2$ leads to a vanishing LLE. Moreover, we observe that the behavior of the LLE dramatically changes as p_0 approaches the origin, raising from zero to a finite value, for velocity values on the first bisectrix intersecting the isostochasticity curve marking the strong stochasticity threshold in Fig. 1. The use of the LLE provides a new signature of the nature of the chaos experienced by the system. One observes that, consistently with Fig. 1, Fig. 2 is not symmetric with respect to the origin: since $\overline{P}$ has been taken positive, strong chaos arises for a larger velocity modulus at the left than at the right of Fig. 2.

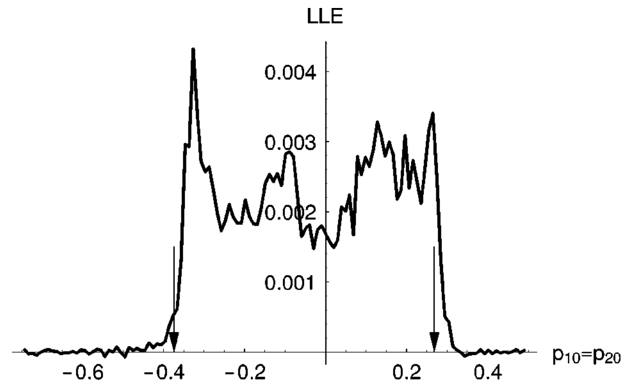

FIG. 2. Largest Lyapunov exponent for the system coupling two particles to one wave. Arrows mark the velocities corresponding to the value of the stochasticity $s=0.55$ as calculated from Eq. (15) on the bisectrix, i.e., with $p_{1r}=p_{2r}$.

IV. LOCALITY OF THE INTERACTION OF *N* PARTICLES WITH ONE WAVE

A. Rough estimate of the velocity borders of stochasticity

The two-particle case cannot emphasize the mean-field nature of the self-consistent potential. When N is large, one can still try to isolate two degrees of freedom among the N degrees of freedom of the full system. In this matter, suppose that these N particles have a large enough extension in velocity (i.e., they form a warm beam) that one can consider among them two "boundary" particles remote from the resonance cat's eye. Their motion will therefore be almost integrable. One can try to estimate the velocity width of the stochastic domain, surrounding the wave resonance, by using the approach developed previously, in a rigorous manner, for the two-particle case.

Let us assume that, within some time interval $[t_0;t_0+\tau]$, the energy of the system may be approximated by

$$H(\mathbf{p},\mathbf{q}) = h_{1,2}(p_1,p_2,q_1,q_2,t_0) + \sum_{l=3}^{N}\left[\frac{p_l^2}{2} - \sqrt{\frac{2\eta}{N}} \times\left(\overline{P} - \sum_{l=3}^{N} p_l - p_1(t_0) - p_2(t_0)\right)^{1/2}\cos q_l\right]$$

with

$$h_{1,2}(p_1,p_2,q_1,q_2,t_0)=\frac{p_1^2}{2}+\frac{p_2^2}{2}-\sqrt{\frac{2\eta}{N}}\left(\overline{P}-\sum_{l=3}^{N}p_l(t_0)-p_1-p_2\right)^{1/2}(\cos q_1+\cos q_2).$$

At $t=t_0$, this separation is exact. This separation will be valid, during a sufficiently long rtime interval τ, provided the variations of the wave amplitude (or, equivalently, of the mean-field velocity coupling) are small, which is consistent with an almost integrable state of the system. As before, in order that the dynamics generated by $h_{1,2}$ be locally reducible to one-and-a-half degrees of freedom, the potential energy term has to be a small pertubation of the kinetic term. We need here the assumption of a sufficiently warm beam ensuring the existence of at least two particles far enough away from the cat's eye. Then the previous procedure applies and gives the stochasticity parameter

$$s(t_0)=4\left(\frac{2\eta}{N}\right)^{1/4}\left(\overline{P}-\sum_{l=3}^{N}p_l(t_0)-p_{1r}-p_{2r}\right)^{1/4}\times\frac{|p_{1r}p_{2r}|}{(p_{1r}^2+p_{2r}^2)^{3/2}}. \tag{22}$$

In the mean-field large N limit, the wave intensity is of order N [21] so that, according to Eq. (5), the following inequalities can be easily satisfied:

$$(2\eta/N)^{1/4}\left(\overline{P}-\sum_{l=3}^{N}p_l(t_0)\right)^{1/4}<|p_{1r}|,|p_{2r}|\ll\overline{P}-\sum_{l=3}^{N}p_l(t_0). \tag{23}$$

One can then define the trapping frequency as

$$\omega_B(t_0)\simeq(2\eta/N)^{1/4}\left(\overline{P}-\sum_{l=3}^{N}p_l(t_0)-p_{1r}-p_{2r}\right)^{1/4}, \tag{24}$$

and one obtains an expression of the stochasticity parameter involving the reference velocities normalized by $\omega_B(t_0)$. Putting $p_{1r}=x\omega_B(t_0)$ and $p_{2r}=y\omega_B(t_0)$, one gets

$$s=4|xy|(x^2+y^2)^{-3/2}, \tag{25}$$

that is, an expression invariant with respect to $x\mapsto -x$ and $y\mapsto -y$ and $(x,y)\mapsto(y,x)$. Figure 3 shows the isostochasticity curve relative to $s_c=0.7$.

This provides a rough qualitative bound on the borders of the stochastic sea: a particle having its velocity a distance of $2\omega_B$ (on spatial average) from the center of the resonance will only feel the wave in a perturbative way. One can thus draw the KAM tori fixing this bound as the constant energy lines having velocities $\pm 2\omega_B$ at the positions of the hyperbolic points. Turning to velocity space representation instead of the energy representation, this means that there should be no particle undergoing strong chaos outside an $8\omega_B$-width domain around the origin.

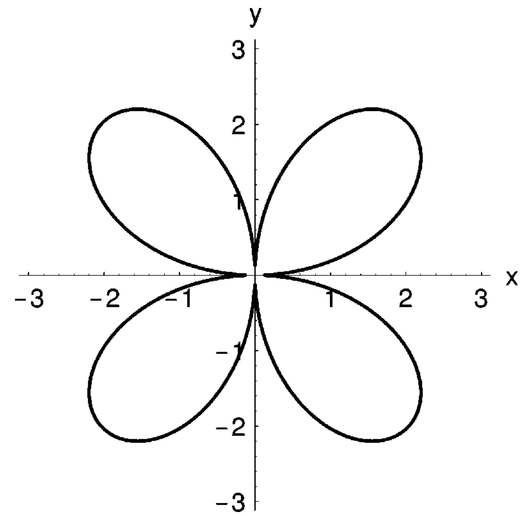

FIG. 3. Curve of isostochasticity $s(x,y)=0.7$ for the reduced quasi-integrable system coupling N particles to one wave.

B. Numerical cartography of the stochasticity in velocity space

We numerically investigate the local stochasticity in velocity space that develops from an initially almost integrable state. At initial time, particles are distributed on velocity beams, uniformly in space, with a flat velocity distribution function, i.e., we start from water-bag initial conditions and the wave has a finite amplitude such that the initial resonance cat's eye has a half width equal to $2\omega_{B0}$. More precisely, we initially distribute the particles according to the distribution function

$$f_0(x,v)=\begin{cases}1/(4\pi\Delta) & \text{if } -\Delta\leqslant v\leqslant\Delta\\ 0 & \text{otherwise,}\end{cases} \tag{26}$$

where the velocity half width Δ is equal to about $12\omega_{B0}$ in the simulations we present. Actually, in order to measure the extension of stochasticity in the velocity space, one has to choose Δ sufficiently large compared to the resonance half width. As the number of particles is finite, the nonexistence of Bernstein-Greene-Kruskal equilibria for finite N [22] ensures that such an initial state cannot be an equilibrium state for the wave-particle system so that this will be slightly destabilized. Then the system naturally evolves due to its intrinsic stochasticity, as any large-N degrees of freedom Hamiltonian system. As for the two-particle system, we computed the finite- (short)-time LLE and measured the number of separatrix crossings by time unit as a function of the initial velocity. These two indicators are displayed in Fig. 4. We checked that identical figures were obtained independently of the (large enough) value of Δ, namely, for $\Delta=25\omega_{B0}$ and for $\Delta=12\omega_{B0}$, as is the case in Fig. 4. The bold dot-dashed curve represents the rescaled number of separatrix crossings by time unit experienced by all the particles having the initial velocity p_0, as a function of p_0, during some initial stage. It

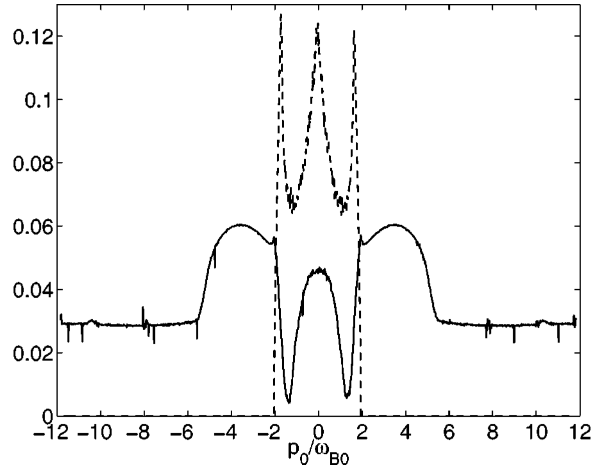

FIG. 4. Finite- (short)-time LLE (thin line) and rescaled number of separatrix crossings by time unit (bold dot-dashed line) as a function of the rescaled initial velocity p_0/ω_{B0}. Please note that, in order to clarify the figure, the LLE curve has been shifted towards the horizontal axis by the value 0.2.

turns out that this number is null for p_0 outside the initial wave resonance, but presents sharp peaks around $-2\omega_{B0}$ and $2\omega_{B0}$. There is a third peak around the zero velocity that can be associated with the particles lying initially close to the x points. This fraction of particles compensates for the trapped particles also having an initially null velocity that shall remain trapped for long times. The percentage of particles experiencing separatrix crossings diminishes as p_0 slightly departs from 0 as the fraction of particles that are trapped for long times increases. Actually, the plot of the mean number of separatrix crossings as a function of initial velocity in Fig. 4 is consistent with the fact that there is an increasing number of particles in the vicinity of the wave separatrix as the velocity increases from 0 up to $2\omega_{B0}$. The relative maximum around $p_0=0$ then signals that the region of the x point is effectively the most active for the trapping/detrapping process. Those particles that undergo frequent trappings and detrappings due to the fluctuations of the wave amplitude may be transported to different velocities as they rotate with the wave. This provides the mechanism for velocity diffusion [23,24].

Let us now consider the results obtained for the LLE as a function of the initial momentum p_0. As this is a finite-time estimate, even the LLE computed around an integrable motion will be nonzero. This just means that the relaxation of the LLE towards zero is still processing. The mean level attained by the LLE relaxing towards zero at the end of the computation is the one exhibited by large-velocity particles (with a modulus of the velocity larger than $5\omega_{B0}$). This provides the level of reference for the LLE. One observes strong correlations between the LLE curve and the curve indicating the number of separatrix crossings as a function of N. The contribution of particles close to the x points explains the relative extremum of the average LLE around $p_0=0$. For particles having a modulus of initial velocity of about $1.5\omega_{B0}$, the LLE is minimal, that is, the average motion experienced by those particles is even more regular than that of particles far away from the resonance that is taken as reference. The main feature of interest provided by the LLE curve is that the chaos is (on spatial average) maximal at the borders of the resonance for particles that do not experience any separatrix crossings. That means that chaos is maximal in velocity regions surrounding the wave resonance where the particles feel the wave but are too distant from the separatrix to be trapped. We can check that this agrees with the qualitative boundary on strong chaos for the N-particle system written previously in the sense that if the modulus of the velocity is larger than $5\omega_{B0}$ there is no longer chaos (only a background finite-time level for the LLE). As the wave intensity fluctuates, expression (25) can only be qualitatively applied.

V. FINAL DISCUSSION

The previous numerical simulations agree with the longstanding picture and understanding of the localization of chaos around a resonance. But this has been almost exclusively constructed up to now from a one or one-and-a-half degrees of freedom prescribed pendulum Hamiltonian. The novelty of the results presented above is due to the large-N self-consistent dynamics: the system evolves naturally from one slightly out-of-equilibrium initial condition under its intrinsic stochasticity and the full self-consistent dynamics reveals the locality of the interaction through the localization of chaos in velocity space. This study shows that the regions of the strongest chaos develop at the borders to the velocity domain swept by the wave resonance. This should be taken into account in the discrimination between resonant particles and nonresonant ones, which can be absorbed in the adiabatic treatment of the background plasma. Cutting the resonant zone just at the wave resonance borders is too drastic a simplification and a larger velocity domain should be considered. Actually, near-resonant particles make a major contribution to the stochasticity of the system and may then be thought to play a leading role in the decoherence process of the wave. These results may be important in improving numerical mixed fluid/kinetic modelings of fusion plasmas dynamics where wave-particle interactions occur, suggesting that a kinetic treatment be extended up to near-resonant velocity domains. This phenomenon may be related to the fact that the oscillations of the trapped particles in the large-amplitude wave trough may destabilize sidebands to the trapping frequency ω_{B0}. These trapped particles have been shown to have a very regular motion so that they can act coherently. This phenomenon has also been observed during the nonlinear stage of the warm beam-wave instability in Ref. [24]. The generation of essentially the lowest harmonics to the trapping frequency contributes to increasing the thermalization of the wave-particle system in the vicinity of the single-wave resonance domain.

This study involves a single-wave model, which differs from the quasilinear approach involving the long-time fate of a continuum or large number of overlapping waves. The single-wave approximation, which is not addressed by the quasilinear theory, has proved to be relevant in a variety of physical situations where the dynamics is effectively dominated by a single mode. For instance, it was realized in recent years that the understanding of the dynamics of a single wave is of essential importance in the containment of charged fusion products in a tokamak. In this spirit, an evolution equation for the wave amplitude of an unstable mode near marginal stability, where resonant wave-particle interaction is balanced by collision relaxation processes, was proposed by the authors of Refs. [25] and later shown to be consistent with experimental data, obtained on the JET tokamak, of kinetic instabilities in Alfvén wave-particle interaction [26].

The above study of the localization of chaos in the single-wave collisionless model complements the usual quasilinear treatment in situations where there is no large-scale transport among the modes and no global diffusion. Rather, this situation reveals an intricate nonlinear behavior localized around the wave resonance that was little studied in the past.

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